

# GRADED DUALITY OF KOSZUL COMPLEXES ASSOCIATED WITH CERTAIN HOMOGENEOUS POLYNOMIALS

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**ABSTRACT.** We show the graded duality of the cohomology groups of the Koszul complexes defined by the partial derivatives of homogeneous polynomials with one-dimensional singular loci, generalizing a well-known result in the isolated singularity case. The top cohomology of the Koszul complex is not necessarily Cohen-Macaulay, but is an extension of Cohen-Macaulay modules of dimension 0 and 1, where the 0-dimensional submodule is graded self-dual as in the isolated singularity case, but the graded dual of the 1-dimensional quotient is the second highest cohomology of the Koszul complex, up to a certain shift of grading. We also give some formulas for the dimensions of their grading.

## Introduction

Let  $f$  be a homogeneous polynomial in  $R := \mathbf{C}[x_1, \dots, x_n]$  with degree  $d$ . Consider the shifted Koszul complex

$$K_f^\bullet := (\Omega^\bullet, df \wedge)[n].$$

Here  $\Omega^j := \Gamma(\mathbf{C}^n, \Omega_{\mathbf{C}^n}^j)$  with  $\Omega_{\mathbf{C}^n}^j$  algebraic so that the  $\Omega^j$  are finite free graded  $R$ -modules, and the degree of  $\Omega^j$  in  $K_f^\bullet$  is shifted. We have more precisely

$$K_f^j = \Omega^{j+n}(jd) \quad (j \in \mathbf{Z}),$$

where the shift of degree by  $p$  of a graded module  $M$  in general is denoted by  $M(p)$ , which is defined by  $M(p)_k = M_{k+p}$ . Since the dualizing complex is given by  $\Omega^n[n]$  in this case, we get the self-duality

$$\mathbf{D}(K_f^\bullet) = K_f^\bullet(nd).$$

Assume

$$\dim \text{Sing } f^{-1}(0) \leq 1.$$

It is well-known (and is easy to show) that this condition implies

$$H^j K_f^\bullet = 0 \quad \text{if } j \neq -1, 0.$$

Set

$$M := H^0 K_f^\bullet, \quad N := H^{-1} K_f^\bullet.$$

Let  $\mathbf{m} = (x_1, \dots, x_n) \subset R$ , the maximal graded ideal. Set

$$M' := H_{\mathbf{m}}^0 M, \quad M'' := M/M'.$$

These are finitely generated graded  $R$ -modules having the decompositions  $M = \bigoplus_k M_k$ , etc. Note that there is a shift of the grading on  $N$  by  $d$  between this paper and [DiSt1], [DiSt2]. Define the (higher) dual graded  $R$ -modules by

$$D_i(M) := \text{Ext}_R^{n-i}(M, \Omega^n) \quad (i \in \mathbf{Z}),$$

and similarly for  $D_i(N)$ , etc. From the above self-duality of the Koszul complex  $K_f^\bullet$ , we can deduce the following duality.

**Theorem 1.** *There are canonical isomorphisms of graded  $R$ -modules*

$$(0.1) \quad \begin{aligned} D_0(M) &= D_0(M') = M'(nd), \\ D_1(M) &= D_1(M'') = N(nd), \\ D_1(N) &= M''(nd), \end{aligned}$$

and  $D_i(M)$ ,  $D_i(M')$ ,  $D_i(M'')$ ,  $D_i(N)$  vanish for other  $i$ .

This implies that  $M'$ ,  $M''$  and  $N$  are Cohen-Macaulay graded  $R$ -modules of dimension 0, 1 and 1 respectively (but  $M$  itself is not Cohen-Macaulay). Moreover  $M'$  is graded self-dual, and  $M''$  and  $N$  are graded dual of each other, up to a shift of grading.

For  $k \in \mathbf{Z}$ , set

$$\mu_k = \dim M_k, \quad \mu'_k = \dim M'_k, \quad \mu''_k = \dim M''_k, \quad \nu_k = \dim N_k.$$

Let  $g := \sum_{i=1}^n x_i^d$ , and  $\gamma_k := \dim(H^n K_g^\bullet)_k = \dim(\Omega^n / \sum_{i=1}^n x_i^{d-1} \Omega^n)_k$ , so that

$$(0.2) \quad \sum_k \gamma_k t^k = (t^d - t)^n / (t - 1)^n.$$

It is known (see [Di], [DiSt1], [DiSt2]) that

$$(0.3) \quad \mu_k = \mu'_k + \mu''_k = \nu_k + \gamma_k \quad (k \in \mathbf{Z}),$$

since the Euler characteristic of a bounded complex is independent of its differential if the components of the complex are finite dimensional.

By the first assertion of (0.1) together with (1.1.4) for  $i = 1$  and by (0.2), we get the following symmetries:

$$\textbf{Corollary 1.} \quad \mu'_k = \mu'_{nd-k}, \quad \gamma_k = \gamma_{nd-k} \quad (k \in \mathbf{Z}).$$

Let  $Z := \{f = 0\} \subset Y := \mathbf{P}^{n-1}$ , and  $\Sigma := \text{Sing } Z$ . Set  $\tau := \sum_{z \in \Sigma} \tau_z$  with  $\tau_z$  the Tjurina number of  $Z$  at  $z \in \Sigma$ , i.e.

$$\tau_z := \dim_{\mathbf{C}} \mathcal{O}_{Y,z} / (h_z, \partial h_z),$$

where  $h_z$  is a local defining equation of  $Z$  at  $z$ , and  $\partial h_z$  is the Jacobian ideal of  $h_z$  generated by its partial derivatives. By Theorem 1,  $M''$  and  $N$  are Cohen-Macaulay, and are dual of each other up to a shift of grading. Combining this with the graded local duality (1.1.4) for  $i = 1$  (see [BH], [Ei], etc.) together with (1.9.3) below, we get the following.

$$\textbf{Corollary 2.} \quad \mu''_k + \nu_{nd-k} = \tau \quad (k \in \mathbf{Z}).$$

Here the calculation of the local cohomology in the local duality is not so trivial (see Remark (1.7) below), and we can also use Schnell's exact sequence as in [Sch], Prop. 2.1 (see also [SaSch]). Note that Corollary 2 can also be deduced from Th. 3.1 in [Di], see Remark (1.9)(i) below. By Corollaries 1 and 2 together with (0.3), we get the following.

$$\textbf{Corollary 3.} \quad \mu'_k = \mu_k + \mu_{nd-k} - \gamma_k - \tau, \quad \mu''_k = \tau - \mu_{nd-k} + \gamma_k \quad (k \in \mathbf{Z}).$$

This means that  $\mu'_k$  and  $\mu''_k$  are essentially determined by  $\mu_k$  and  $\mu_{nd-k}$ . As is seen later (see e.g. (0.6)),  $\mu''_k$ ,  $\nu_k$  are weakly increasing sequences of non-negative integers. Assuming  $\text{Sing } Z \neq \emptyset$ , we have  $\mu''_k = \nu_k = \tau > 0$  for  $k \gg 0$ , hence  $M''$ ,  $N$  are nonzero, although  $M'$  may vanish, see Remark (1.9)(iii) below. By Corollary 2 and (0.3) we get the following.

$$\textbf{Corollary 4.} \quad \gamma_k - \mu'_k = \mu''_k + \mu''_{nd-k} - \tau \quad (k \in \mathbf{Z}).$$

Here a fundamental question seems to be the following.

**Question 1.** Are both sides of the above equality non-negative?

This seems to be closely related to the subject treated in [CD], [Di], [DiSt1], [DiSt2], etc. Set

$$\rho_k := \min(\mu_k'', \mu_{nd-k}''), \quad \sigma_k := \max(\mu_k'', \mu_{nd-k}'') \quad (k \in \mathbf{Z}),$$

so that we have the symmetries:

$$\rho_k = \rho_{nd-k}, \quad \sigma_k = \sigma_{nd-k} \quad (k \in \mathbf{Z}).$$

By Corollary 4, we get

$$\gamma_k - \mu_k' = \rho_k + \sigma_k - \tau \quad (k \in \mathbf{Z}).$$

Note that the support of the functions  $\gamma_k$ ,  $\mu_k'$ ,  $\rho_k$  of  $k \in \mathbf{Z}$  are contained in  $[n, n(d-1)]$ . A more concrete question would be the following.

**Question 2.** When does the following hold?

$$(0.4) \quad \gamma_k - \mu_k' = \rho_k, \quad \text{i.e.} \quad \sigma_k = \tau \quad \text{for any } k \in \mathbf{Z}.$$

By the definition of  $\sigma_k$ , and using Corollary 2, we see that (0.4) is equivalent to

$$(0.5) \quad \mu_k'' = \tau \quad \text{for } k \geq nd/2, \quad \text{i.e.} \quad \nu_k = 0 \quad \text{for } k \leq nd/2.$$

By the definition of  $N$ , the last condition in (0.5) cannot hold if there is a nontrivial relation of very low degree between the partial derivatives of  $f$ , e.g. in case  $f$  is a polynomial of  $n-1$  variables (or close to it), see Remark (3.3) below. However, it holds in relatively simple cases, including the nodal hypersurface case by [DiSt2], Th.2.1, see Remark (3.4) below.

Let  $y := \sum_{i=1}^n c_i x_i$  with  $c_i \in \mathbf{C}$  sufficiently general so that  $\{y=0\} \subset \mathbf{C}^n$  is transversal to any irreducible component of  $\text{Sing } f^{-1}(0)$ . Then  $M''$  and  $N$  are finitely generated torsion-free graded  $\mathbf{C}[y]$ -modules of rank  $\tau$ , where  $\tau$  is as in Corollary 2. Let  $u_i, v_i$  be their free homogeneous generators ( $i \in [1, \tau]$ ). Set

$$a_i = \deg u_i, \quad b_i = \deg v_i \quad (i \in [1, \tau]),$$

where we may assume that  $(a_i)$  and  $(b_i)$  are increasing sequences in a weak sense, i.e.  $a_i \leq a_{i+1}$ , etc. These numbers are independent of the choice of the generators  $u_i, v_i$ , since

$$(0.6) \quad \begin{aligned} \#\{i \mid a_i = k\} &= \mu_k'' - \mu_{k-1}'', \\ \#\{i \mid b_i = k\} &= \nu_k - \nu_{k-1}. \end{aligned}$$

By Corollary 2 we get the following.

**Corollary 5.**  $a_i + b_{\tau+1-i} = nd + 1 \quad (i \in [1, \tau]).$

Using the graded ideal of  $R$  corresponding to a certain subset  $\Sigma'$  of  $\Sigma = \text{Sing } Z$  where  $Z = \{f=0\} \subset \mathbf{P}^{n-1}$  as above, we can show the following:

**Proposition 1.** *Let  $r$  be the dimension of the vector subspace of  $\mathbf{C}^n$  generated by the one-dimensional vector subspaces corresponding to the singular points of  $Z$ . Then*

$$a_1 = n, \quad a_i = n + 1 \quad (i \in [2, r]).$$

In certain simple cases, this determines the  $a_i$ , and hence the  $b_i$  by Corollary 5, and also the  $\mu_k', \mu_k'', \nu_k$  by (0.3) and (0.6) as follows.

**Corollary 6.** *Assume  $Z$  has only ordinary double points  $z_1, \dots, z_\tau$ , and moreover the  $z_i$  correspond to linearly independent vectors in  $\mathbf{C}^n$  so that  $\tau = r \leq n$ . Then*

$$a_1 = n, \quad a_i = n + 1 \quad (i \in [2, \tau]),$$

hence

$$\begin{aligned} \mu_k'' &= 0 \quad (k < n), \quad \mu_n'' = 1, \quad \mu_k'' = \tau \quad (k > n) \\ \nu_k &= 0 \quad (k < n(d-1)), \quad \nu_{n(d-1)} = \tau - 1, \quad \nu_k = \tau \quad (k > n(d-1)), \end{aligned}$$

and furthermore

$$\mu_k' = 0 \text{ if } k \notin (n, n(d-1)), \quad \mu_k' = \gamma_k - \tau \text{ if } k \in (n, n(d-1)),$$

where  $(n, n(d-1)) \subset \mathbf{R}$  denotes an open interval.

Note that this can also be deduced from the results in [Di]. The situation becomes, however, rather complicated if the number of singular points is quite large, see [CD], [Di], [DiSt1], [DiSt2]. Set

$$S(\mu) := \sum_k \mu_k t^k \in \mathbf{Z}[[t]],$$

and similarly for  $S(\mu'_k)$ , etc. (Note that  $\mu_k$ , etc. vanish for  $k < 0$ .) In certain cases we can apply a Thom-Sebastiani type theorem as follows.

**Proposition 2.** *Assume  $f = f_1 + f_2$  with  $f_1 \in \mathbf{C}[x_1, \dots, x_{n_1}]$ ,  $f_2 \in \mathbf{C}[x_{n_1+1}, \dots, x_n]$  where  $1 < n_1 < n - 1$ . If the dimensions of the singular loci of  $f_1^{-1}(0) \subset \mathbf{C}^{n_1}$  and  $f_2^{-1}(0) \subset \mathbf{C}^{n-n_1}$  are respectively 1 and 0, then there are isomorphisms of graded  $R$ -modules*

$$M' = M'_{(1)} \otimes_{\mathbf{C}} M'_{(2)}, \quad M'' = M''_{(1)} \otimes_{\mathbf{C}} M'_{(2)}, \quad N = N_{(1)} \otimes_{\mathbf{C}} M'_{(2)},$$

hence we have the equalities

$$S(\mu') = S(\mu'_{(1)}) S(\mu'_{(2)}), \quad S(\mu'') = S(\mu''_{(1)}) S(\mu'_{(2)}), \quad S(\nu) = S(\nu_{(1)}) S(\mu'_{(2)}),$$

where  $M'_{(i)}$ ,  $M''_{(i)}$ ,  $N_{(i)}$ , and  $\mu'_{(i),k}$ ,  $\mu''_{(i),k}$ ,  $\nu_{(i),k}$  ( $k \in \mathbf{Z}$ ) are defined for  $f_i$  ( $i = 1, 2$ ).

Here one cannot assume that the singular locus of  $f_2^{-1}(0)$  is also 1-dimensional since that of  $f^{-1}(0)$  is the product of those of  $f_1^{-1}(0)$  and  $f_2^{-1}(0)$ . Note that  $\mu'_{(2),k}$  is given by (0.2). In case  $n_1 = 2$ , we can calculate  $\mu'_{(1),k}$ ,  $\mu''_{(1),k}$ ,  $\nu_{(1),k}$  for  $f_1$  by Lemma (2.3) below, and get the following.

**Theorem 2.** *With the notation and the assumption of Proposition 2, assume  $n_1 = 2$ . Let  $r$  be the number of the irreducible components of  $f_1^{-1}(0) \subset \mathbf{C}^2$ . For  $a \in \mathbf{N}$ ,  $b \in \mathbf{N} \cup \{+\infty\}$ , set*

$$S(a, b) := \sum_{k=a}^b t^k \in \mathbf{Z}[[t]] \text{ if } a \leq b, \text{ and } 0 \text{ otherwise.}$$

Then

$$\begin{aligned} S(\mu') &= S(1, r-1) S(d-r+1, d-1) S(1, d-1)^{n-2}, \\ S(\mu'') &= S(1, +\infty) S(1, d-r) S(1, d-1)^{n-2}, \\ S(\nu) &= S(d+r-2, +\infty) S(1, d-r) S(1, d-1)^{n-2}. \end{aligned}$$

Here the relation between  $S(\mu')$  and the spectrum of  $f$  (see [St2]) is not yet clear. This seems to be an interesting question, see Remark (2.7) below. In the isolated singularity case,  $S(\mu')$  coincides with the spectrum of  $f$  after replacing  $t$  with  $t^{1/d}$ , see [St1].

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In Section 1 we prove Theorem 1 after reviewing some commutative algebra for the convenience of the reader. In Section 2 we prove Theorem 2 after showing Propositions 1 and 2. In Section 3 we give some examples and remarks.

## 1. Proof of Theorem 1

In this section we prove Theorem 1 after reviewing some commutative algebra for the convenience of the reader.

**1.1. Graded local duality.** Let  $R = \mathbf{C}[x_1, \dots, x_n]$ , and  $\mathbf{m} = (x_1, \dots, x_n) \subset R$ . Set

$$(1.1.1) \quad \Omega^k = \Gamma(\mathbf{C}^n, \Omega_{\mathbf{C}^n}^k) \quad (k \in \mathbf{Z}).$$

Here  $\Omega_{\mathbf{C}^n}^k$  is algebraic, and  $\Omega^k$  is a finite free graded  $R$ -module with  $\deg x_i = \deg dx_i = 1$ .

For a bounded complex of finitely generated graded  $R$ -modules  $M^\bullet$ , define

$$(1.1.2) \quad \begin{aligned} D_i(M^\bullet) &:= \text{Ext}_R^{n-i}(M^\bullet, \Omega^n) = H^{-i}(\mathbf{D}(M^\bullet)) \\ \text{with } \mathbf{D}(M^\bullet) &:= \mathbf{R}\text{Hom}_R(M^\bullet, \Omega^n[n]), \end{aligned}$$

where  $\mathbf{D}(M^\bullet)$  can be defined by taking a graded free resolution  $P^\bullet \rightarrow M^\bullet$ .

For a finitely generated graded  $R$ -module  $M$ , set

$$(1.1.3) \quad H_{\mathbf{m}}^0 M := \{a \in M \mid \mathbf{m}^k a = 0 \text{ for } k \gg 0\}.$$

Let  $H_{\mathbf{m}}^i M$  be the cohomological right derived functors ( $i \in \mathbf{N}$ ). These are defined by taking a graded injective resolution of  $M$ . We can calculate them by taking a graded free resolution of  $M$  as is explained in textbooks of commutative algebra, see e.g. [BH], [Ei]. Indeed,  $H_{\mathbf{m}}^i R = 0$  for  $i \neq n$ , and

$$H_{\mathbf{m}}^n R = \mathbf{C}[x_1^{-1}, \dots, x_n^{-1}]_{x_1 \dots x_n},$$

where the right-hand side is identified with a quotient of the graded localization of  $R$  by  $x_1 \cdots x_n$ . We then get the *graded local duality* for finitely generated graded  $R$ -modules  $M$ :

$$(1.1.4) \quad D_i(M)_k = \text{Hom}_{\mathbf{C}}((H_{\mathbf{m}}^i M)_{-k}, \mathbf{C}) \quad (k \in \mathbf{Z}, i \geq 0),$$

see loc. cit. (Indeed, this can be reduced to the case  $M = R$  by the above argument.)

**1.2. Remarks.** (i) The functors  $H_{\mathbf{m}}^i$  and  $D_i$  are compatible with the corresponding functors for non-graded  $R$ -modules under the forgetful functor, and moreover, the latter functors are compatible with the corresponding sheaf-theoretic functors as is well-known in textbooks of algebraic geometry, see e.g. [Ha]. However, the information of the grading is lost by passing to the corresponding sheaf unless we use a sheaf with  $\mathbf{C}^*$ -action.

(ii) If  $M$  is a finitely generated graded  $R$ -module, then it is well-known that

$$(1.2.1) \quad D_i(M) = 0 \quad \text{for } i < 0.$$

**1.3. Spectral sequences.** For a bounded complex of finitely generated graded  $R$ -modules  $M^\bullet$ , we have a spectral sequence

$$(1.3.1) \quad {}^p E_2^{p,q}(M^\bullet) = D_{-p}(H^{-q} M^\bullet) \implies D_{-p-q}(M^\bullet).$$

This can be defined for instance by taking graded free resolutions of  $H^i M^\bullet$  and  $\text{Im } d^i$  for  $i \in \mathbf{Z}$ , and then extending these to a graded free resolution of  $M^\bullet$  by using the short exact sequences

$$0 \rightarrow \text{Im } d^{i-1} \rightarrow \text{Ker } d^i \rightarrow H^i M^\bullet \rightarrow 0, \quad 0 \rightarrow \text{Ker } d^i \rightarrow M^i \rightarrow \text{Im } d^i \rightarrow 0,$$

as is explained in classical books about spectral sequences. We can also construct (1.3.1) by using the filtration  $\tau_{\leq -q}$  on  $M^\bullet$  as in [De].

Applying (1.3.1) to  $\mathbf{D}(M^\bullet)$  and using  $\mathbf{D}(\mathbf{D}(M^\bullet)) = M^\bullet$ , we get

$$(1.3.2) \quad {}^p E_2^{p,q}(M^\bullet) = D_{-p}(D_q(M^\bullet)) \implies H^{p+q} M^\bullet.$$

**1.4. Lemma.** *Let  $\mathcal{S}h(M)$  denote the coherent sheaf on  $X := \mathbf{C}^n$  corresponding to a finitely generated graded  $R$ -module  $M$ . Then we have the following equivalence.*

$$\begin{aligned}
 (1.4.1) \quad H_{\mathbf{m}}^0 M = M &\iff \text{supp } \mathcal{S}h(M) \subset \{0\} \\
 &\iff M \text{ is finite dimensional over } \mathbf{C}, \\
 &\iff D_i(M) = 0 \text{ for any } i \neq 0.
 \end{aligned}$$

*Proof.* This is almost trivial except possibly for the last equivalence. It can be shown by restricting to a sufficiently general point of the support of  $\mathcal{S}h(M)$  in case the support has positive dimension. Here we use the fact that the dual  $\mathbf{D}(\mathcal{S}h(M))$  is compatible with the direct image under a closed embedding, and this follows from Grothendieck duality for closed embeddings as is well-known, see e.g. [Ha]. This finishes the proof of Lemma (1.4).

The following is well-known, see [BH], [Ei], etc. We note here a short proof for the convenience of the reader.

**1.5. Proposition.** *Let  $M$  be a finitely generated  $R$ -module. Set  $m := \dim \text{supp } \mathcal{S}h(M)$ . Then*

$$(1.5.1) \quad D_i(M) = 0 \text{ for } i > m.$$

*Proof.* There is a complete intersection  $Z$  of dimension  $m$  in  $X = \text{Spec } R$  such that  $M$  is annihilated by the ideal  $I_Z$  of  $Z$ , i.e.  $M$  is an  $R_Z$ -module with  $R_Z := R/I_Z$ , and  $I_Z$  is generated by a regular sequence  $(g_i)_{i \in [1, n-m]}$  of  $R$  with  $g_i M = 0$ . (Here  $M$  is not assumed graded.) Set

$$\omega_Z = \text{Ext}_R^{n-m}(R_Z, \Omega^n).$$

This is called the canonical (or dualizing) module of  $Z$ . We then get

$$(1.5.2) \quad D_i(M) = \text{Ext}_{R_Z}^{-i}(M, \omega_Z[m]),$$

by Grothendieck duality for the closed embedding  $i_Z : Z \hookrightarrow X$ , see e.g. [Ha], etc. In fact, taking an injective resolution  $G$  of  $\Omega^n[n]$ , one can show (1.5.2) by using the canonical isomorphism

$$\text{Hom}_{R_Z}(M, \text{Hom}_R(R_Z, G)) = \text{Hom}_R(M, G).$$

Since the right-hand side of (1.5.2) vanishes for  $i > m$ , the assertion follows.

**1.6. Corollary.** *Let  $M$  be a finitely generated graded  $R$ -module with  $\dim \text{supp } \mathcal{S}h(M) = 1$ . Then we have a short exact sequence*

$$(1.6.1) \quad 0 \rightarrow D_0(D_0(M)) \rightarrow M \rightarrow D_1(D_1(M)) \rightarrow 0,$$

together with

$$(1.6.2) \quad D_0(D_1(M)) = 0, \quad D_1(D_0(M)) = 0.$$

*Proof.* By Lemma (1.5) we get

$${}''E_2^{p,q}(M) = 0 \quad \text{if } (p, q) \notin [-1, 0] \times [0, 1].$$

So the spectral sequence (1.3.2) degenerates at  $E_2$  in this case, and the assertion follows.

**1.7. Remark.** Let  $M$  be a graded  $R$ -module of dimension 1, i.e.  $C := \text{supp } \mathcal{S}h(M)$  is one-dimensional. Let  $I_M \subset R$  be the annihilator of  $M$ . Set  $\bar{R} := R/I_M$ . Let  $y \in R$  be a general element of degree 1 whose restriction to any irreducible component of  $C$  is nonzero. Set  $R' := \mathbf{C}[y] \subset R$ . Let  $\bar{\mathbf{m}}, \mathbf{m}'$  be the maximal graded ideals of  $\bar{R}, R'$ . Let  $H_{(R, \mathbf{m})}^i M$

denote  $H_{\mathbf{m}}^i M$ , and similarly for  $H_{(\bar{R}, \bar{\mathbf{m}})}^i M$ , etc. (to avoid any confusion). There are canonical morphisms

$$(R, \mathbf{m}) \rightarrow (\bar{R}, \bar{\mathbf{m}}) \leftarrow (R', \mathbf{m}'),$$

and they imply canonical morphisms

$$(1.7.1) \quad H_{(R, \mathbf{m})}^i M \leftarrow H_{(\bar{R}, \bar{\mathbf{m}})}^i M \rightarrow H_{(R', \mathbf{m}')}^i M.$$

Indeed, any graded injective resolution of  $M$  over  $\bar{R}$  can be viewed as a quasi-isomorphism over  $R$  or  $R'$ , and we can further take its graded injective resolution over  $R$  or  $R'$ , which induces the above morphisms.

These morphisms are isomorphisms since they are isomorphisms by forgetting the grading as is well-known. (Note that the morphisms  $\text{Spec } R \leftarrow \text{Spec } \bar{R} \rightarrow \text{Spec } R'$  are proper.) Using the long exact sequence associated with the local cohomology and the localization, we can show

$$(1.7.2) \quad H_{(R', \mathbf{m}')}^1 M = M[y^{-1}]/M.$$

So we get the following canonical isomorphism (as graded  $R'$ -modules):

$$(1.7.3) \quad H_{\mathbf{m}}^1 M = M[y^{-1}]/M.$$

This also follows from Schnell's exact sequence in [Sch], Prop. 2.1 (see also [SaSch]).

**1.8. Proof of Theorem 1.** As is explained in the introduction, we have the self-duality

$$\mathbf{D}(K_f^\bullet) = K_f^\bullet(nd),$$

which implies the isomorphisms of graded  $R$ -modules

$$(1.8.1) \quad D_i(K_f^\bullet) = H^{-i} K_f^\bullet(nd).$$

Consider the spectral sequence (1.3.1) for  $M^\bullet = K_f^\bullet$ . By Lemma (1.5) applied to  $M, N$ , this degenerates at  $E_2$ . Combining this with (1.8.1), we thus get

$$(1.8.2) \quad D_1(M) = N(nd), \quad D_0(N) = 0,$$

together with a short exact sequence

$$(1.8.3) \quad 0 \rightarrow D_0(M) \rightarrow M(nd) \rightarrow D_1(N) \rightarrow 0.$$

By (1.6.2) in Corollary (1.6) and Lemma (1.5) applied to  $M, N$ , the proof of Theorem 1 is then reduced to showing that (1.8.3) is naturally identified, up to the shift of grading by  $nd$ , with

$$(1.8.4) \quad 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0.$$

For this, it is enough to show

$$(1.8.5) \quad H_{\mathbf{m}}^0 D_0(M) = D_0(M), \quad H_{\mathbf{m}}^0 D_1(N) = 0.$$

However, the first equality is equivalent to the vanishing of  $D_i(D_0(M))$  for  $i \neq 0$  by Lemma (1.4), and follows from (1.6.2) in Corollary (1.6) together with Lemma (1.5) applied to  $D_0(M)$ . The second equality follows for instance from the local duality (1.1.4) for  $i = 0$  together with (1.6.2) in Corollary (1.6) applied to  $N$ . Thus (1.8.5) is proved. This finishes the proof of Theorem 1.

**1.9. Remarks.** (i) Corollary 2 can also be deduced from [Di], Th. 3.1. Indeed, by the argument in Section 2 in loc. cit., we can deduce

$$(1.9.1) \quad \text{def}_{k-n} \Sigma_f = \tau - \mu_k'',$$

where  $\text{def}_{k-n}\Sigma_f$  is as in loc. cit. Moreover, Th. 3.1 in loc. cit. gives

$$(1.9.2) \quad \text{def}_{k-n}\Sigma_f = \mu_{nd-k} - \gamma_{nd-k} = \nu_{nd-k}.$$

So Corollary 2 follows.

(ii) It is well-known that

$$(1.9.3) \quad \dim_{\mathbf{C}} M_k'' = \dim_{\mathbf{C}} M_k = \tau \quad \text{if } k \gg 0.$$

Indeed, the first equality of (1.9.3) is trivial, and it is enough to show the last equality. Changing the coordinates, we may assume  $x_n = y$ , where  $y$  is as in the introduction. On  $\{x_n \neq 0\} \subset \mathbf{C}^n$ , we have the coordinates  $x'_1, \dots, x'_n$  defined by  $x'_j = x_j/x_n$  for  $j \neq n$ , and  $x'_n = x_n$ . Using these, we have  $f(x) = x_n^d h(x')$ , where  $x' = (x'_1, \dots, x'_{n-1})$ . This implies that the restriction of  $\mathcal{S}h(M)$  to the generic point of an irreducible component of the support of  $M$  corresponding to  $z \in Z$  has rank  $\tau_z$  in the notation of the introduction. So (1.9.3) follows.

(iii) Assume  $\dim \text{Sing } f^{-1}(0) = 1$ , i.e.  $\Sigma = \text{Sing } Z \neq \emptyset$ . Let  $(\partial f) \subset R$  denote the Jacobian ideal of  $f$  (generated by the partial derivatives  $\partial f / \partial x_i$  of  $f$ ). Then the Jacobian ring  $R/(\partial f)$  (which is isomorphic to  $M$  as a graded  $R$ -module up to a shift of grading) is a Cohen-Macaulay ring if and only if  $M' = 0$ . Indeed, these are both equivalent to the condition that  $M$  is a Cohen-Macaulay  $R$ -module (since  $\tau \neq 0$  and hence  $M'' \neq 0$ ). Here Grothendieck duality for closed embeddings is used to show the equivalence with the condition that  $R/(\partial f)$  is a Cohen-Macaulay ring. Note that  $M'$  might vanish in general, for instance if  $f$  is as in Example (3.1) below or even in case  $f = xyz$ .

(iv) Assume  $\bigcap_{i=1}^m g_i^{-1}(0) \subset \mathbf{C}^n$  has codimension  $\geq r$ , where  $g_i \in R$  ( $i \in [1, m]$ ). Then there is a regular sequence  $(h_j)_{j \in [1, r]}$  of  $R$  with  $h_j \in V := \sum_{i=1}^m \mathbf{C} g_i$  by increasing induction on  $r$  or  $m$ . This implies the vanishing of the cohomology of the Koszul complex:

$$H^k K^\bullet(R; g_1, \dots, g_m) = 0 \quad (k < r),$$

by using the  $n$ -ple complex structure of the Koszul complex as is well-known (see Remark (v) below). In fact, we can replace the basis  $(g_i)$  of the vector space  $V$  so that a different expression of the Koszul complex can be obtained. (However, it is not always possible to choose  $h_j$  so that  $\sum_{i=1}^m Rg_i = \sum_{j=1}^r Rh_j$  even if  $\bigcap_{i=1}^m g_i^{-1}(0)$  has pure codimension  $r$  unless  $(g_i)$  is already a regular sequence, i.e.  $r = m$ .)

(v) For  $g_i \in R$  ( $i \in [1, m]$ ), the Koszul complex  $K^\bullet(R; g_1, \dots, g_m)$  can be identified with the associated single complex of the  $m$ -ple complex whose  $(j_1, \dots, j_m)$ -component is  $R$  for  $(j_1, \dots, j_m) \in [0, 1]^m$ , and 0 otherwise, where its  $i$ -th differential  $d_i$  is defined by the multiplication by  $g_i$  on  $R$ .

(vi) Theorem 1 holds with  $df$  in the definition of the Koszul complex replaced by a 1-form  $\omega = \sum_{i=1}^n g_i dx_i$  if the  $g_i$  are homogeneous polynomials of degree  $d - 1$  such that  $\bigcap_i g_i^{-1}(0) \subset \mathbf{C}^n$  is at most 1-dimensional.

## 2. Proof of Theorem 2

In this section we prove Theorem 2 after showing Propositions 1 and 2.

**2.1. Proof of Proposition 1.** Let  $\Sigma'$  be a subset of  $\Sigma (= \text{Sing } Z)$  corresponding to linearly independent  $r$  vectors of  $\mathbf{C}^n$ . Let  $I_{\Sigma'}$  be the (reduced) graded ideal of  $R$  corresponding to  $\Sigma'$ . There is a canonical surjection

$$(2.1.1) \quad M \rightarrow \overline{M} := \Omega^n / I_{\Sigma'} \Omega^n.$$



The target is a graded free  $\mathbf{C}[y]$ -module of rank  $r$ , where  $y$  is as in the introduction, and it has free homogeneous generators  $w_i$  ( $i \in [1, r]$ ) with  $\deg w_1 = n$  and  $\deg w_i = n + 1$  for  $i > 1$ . So the morphism (2.1.1) factors through  $M''$ , and the assertion follows.

**2.2. Proof of Proposition 2.** Using the  $n$ -ple complex structure of the Koszul complex as in Remark (1.9)(v), we get the canonical isomorphism

$$K_f^\bullet = K_{f_1}^\bullet \otimes_{\mathbf{C}} K_{f_2}^\bullet,$$

where  $K_{f_1}^\bullet$  is defined by using the subring  $\mathbf{C}[x_1, \dots, x_{n_1}]$ , and similarly for  $K_{f_2}^\bullet$ . Since  $f_2^{-1}(0)$  has an isolated singularity,  $K_{f_2}^\bullet$  is naturally quasi-isomorphic to  $M'_{(2)}$ . So the assertion follows.

For the proof of Theorem 2, we need the following lemma. Essentially this may be viewed as a special case of Prop. 13 in [CD], see Remark (2.5) below. We note here a short proof of the lemma using Corollaries 1 and 2 and (0.3) for the convenience of the reader.

**2.3. Lemma.** *Assume  $n = 2$ . Let  $r$  be the number of the irreducible components of  $f^{-1}(0) \subset \mathbf{C}^2$ . Then  $\tau = d - r$ , and*

$$\begin{aligned} \mu_k'' &= (k - 1)_{[0, \tau]}, \quad \nu_k = (k - d - r + 1)_{[0, \tau]}, \\ \mu_k' &= \max(r - 1 - |d - k|, 0) \quad (k \in \mathbf{Z}), \end{aligned}$$

where  $C_{[0, \tau]}$  is equal to  $C$  if  $C \in [0, \tau]$ , 0 if  $C < 0$ , and  $\tau$  if  $C > \tau$ .

*Proof.* We have the decomposition

$$f = \prod_{i=1}^r g_i^{m_i},$$

with  $\deg g_i = 1$  and  $m_i \geq 1$ . For  $z \in \mathbf{P}^1$  corresponding to  $g_i^{-1}(0) \subset \mathbf{C}^2$ , we have

$$\tau_z = m_i - 1, \quad \text{and hence} \quad \tau = d - r.$$

Setting

$$f' := \prod_{i=1}^r g_i^{m_i-1},$$

we get

$$M'' = \Omega^2 / f' \Omega^2.$$

Indeed, the right-hand side is a quotient graded  $R$ -module of  $M$ , and is a free graded  $\mathbf{C}[y]$ -module of rank  $\tau$ . Since  $\deg f' = \tau$ , this implies

$$\mu_k'' = (k - 1)_{[0, \tau]}.$$

Using Corollary 2, we then get

$$\nu_k = d - r - (2d - k - 1)_{[0, \tau]} = (k - d - r + 1)_{[0, \tau]}.$$

Here note that

$$\nu_k = 0 \quad \text{if } k \leq d.$$

For  $n = 2$  and  $k \leq d$ , we have

$$\gamma_k = \max(k - 1, 0).$$

By (0.3) we then get for  $k \leq d$

$$\mu_k' = \gamma_k - \mu_k'' = \max(k - 1 - \tau, 0).$$

The formula for  $k \geq d$  follows by using the symmetry in Corollary 1. This finishes the proof of Lemma (2.3).

By an easy calculation we see that Lemma (2.3) is equivalent to the following.

**2.4. Corollary.** *With the notation and the assumption of Lemma (2.3), we have*

$$(2.4.1) \quad \begin{aligned} S(\mu') &= S(1, r-1) S(d-r+1, d-1), \\ S(\mu'') &= S(1, +\infty) S(1, d-r), \\ S(\nu) &= S(d+r-2, +\infty) S(1, d-r), \end{aligned}$$

where  $S(\mu')$ , etc. are as in Theorem 2.

**2.5. Remark.** With the notation and the assumption of Corollary (2.4), the following is shown in [Di], Example 14 (i) as a corollary of Prop. 13 in loc. cit.

$$(2.5.1) \quad S(\mu) = t^2(1 - 2t^{d-1} + t^{d+r-2})/(1-t)^2.$$

By Corollaries 2 and 3 together with (0.3), this should be essentially equivalent to the equalities in (2.4.1). In fact, it seems rather easy to deduce (2.5.1) from (2.4.1). For the converse some calculation seems to be needed. (The details are left to the reader.)

**2.6. Proof of Theorem 2.** The assertion follows from Corollary (2.4) and Proposition 2, since  $S(\mu')$  in the isolated singularity case is invariant by  $\mu$ -constant deformation, and is given by (0.2).

**2.7. Remark.** It seems that  $M'$  is closely related to the spectrum of  $f$  (see [St2]). For the moment it is not clear whether any  $\omega \in M'$  contributes to the spectrum of  $f$  as in the isolated singularity case. (For a relation between  $M''$  and the spectrum in a special case, see Remark (3.5) below.)

### 3. Examples and Remarks.

In this section we give some examples and remarks.

**3.1. Example.** Let  $f$  be as in Theorem 1, and assume further

$$f \in \mathbf{C}[x_1, \dots, x_{n-1}] \subset \mathbf{C}[x_1, \dots, x_n].$$

Then  $f$  has an isolated singularity at the origin of  $\mathbf{C}^{n-1}$ . Set

$$\gamma'_j := \dim_{\mathbf{C}}(\Omega'^{n-1}/df \wedge \Omega'^{n-1})_j \quad \text{with} \quad \Omega'^k := \Gamma(\mathbf{C}^{n-1}, \Omega_{\mathbf{C}^{n-1}}^k).$$

We have the symmetry

$$(3.1.1) \quad \gamma'_j = \gamma'_{(n-1)d-j}.$$

In this case, we have  $M' = 0$ , and

$$(3.1.2) \quad \mu_k = \mu''_k = \sum_{j \leq k-1} \gamma'_j, \quad \nu_k = \sum_{j \leq k-d} \gamma'_j = \sum_{j \geq nd-k} \gamma'_j,$$

where the last equality follows from the symmetry (3.1.1), and Corollary 2 is verified directly in this case.

Equivalently,  $\mu''_k = \mu_k$  and  $\nu_k$  are given as follows:

$$(3.1.3) \quad \begin{aligned} S(\mu) &= S(1, +\infty) S(1, d-1)^{n-1}, \\ S(\nu) &= S(d, +\infty) S(1, d-1)^{n-1}, \end{aligned}$$

where  $S(\mu)$ , etc. are as in Theorem 2, and the order of the coordinates are changed.

**3.2. Example.** Assume  $n, d \geq 3$ . Let

$$(3.2.1) \quad f = x_1^a x_2^{d-a} + \sum_{i=3}^n x_i^d \quad \text{with} \quad 0 < a < d.$$

We can apply Theorems 2 and 3 to this example. More precisely, the calculation of  $\mu'_k$ ,  $\mu''_k$  and  $\nu_k$  are reduced to the case  $n = 2$  by Proposition 2, where  $n_1 = 2$  and

$$f_1 = x_1^a x_2^{d-a}, \quad f_2 = \sum_{i=3}^n x_i^d.$$

The last case is treated by Lemma (2.3) where  $r = 2$ . We get for instance

$$\mu'_{(1),k} = \begin{cases} 1 & \text{if } k = d, \\ 0 & \text{if } k \neq d, \end{cases}$$

and hence

$$\mu'_k = \mu'_{(2),k+d} = \gamma''_{k+d} \quad (k \in \mathbf{Z}),$$

where  $\gamma''_k$  is as in (0.2) with  $n$  replaced by  $n - 2$ . By Theorem 2, we have

$$\begin{aligned} (3.2.2) \quad S(\mu') &= t^d S(1, d-1)^{n-2}, \\ S(\mu'') &= S(1, +\infty) S(1, d-2) S(1, d-1)^{n-2}, \\ S(\nu) &= S(d, +\infty) S(1, d-2) S(1, d-1)^{n-2}, \end{aligned}$$

where  $S(\mu')$ , etc. are as in Theorem 2.

**3.3. Remark.** If there is a nontrivial relation of degree  $k \leq d - 2$  among the partial derivatives  $f_i := \partial f / \partial x_i$ , i.e. if there are homogeneous polynomials  $g_i$  of degree  $k \leq d - 2$  with  $\sum_i g_i f_i = 0$  and  $g_i \neq 0$  for some  $i$ , then we have

$$(3.3.1) \quad \nu_{d+n+k-1} \neq 0,$$

and hence

$$(3.3.2) \quad \text{Conditions (0.4-5) do not hold if } (n-2)(d-2) \geq 2(k+1).$$

Indeed, (3.3.1) follows from the definition  $N := H^{-1}K_f^\bullet$  since  $\deg f_i = d - 1$ .

This applies to  $f$  in Example (3.1) with  $k = 0$  since  $f_n = 0$ , and to  $f$  in Example (3.2) with  $k = 1$  since

$$(d-a)x_1 f_1 = ax_2 f_2.$$

**3.4. Remark.** Conditions (0.4-5) hold in the nodal hypersurface case by [DiSt2], Th.2.1. Indeed, it is shown there that

$$(3.4.1) \quad \nu_k = 0 \text{ if } k \leq (n_1 + 1)d \text{ with } n \text{ even or } k \leq (n_1 + 1)d - 1 \text{ with } n \text{ odd,}$$

where  $n_1 := [(n-1)/2]$ . (There is a difference in the grading on  $N$  by  $d$  between this paper and loc. cit., and  $n$  in this paper is  $n + 1$  in loc. cit.) Note that this implies also

$$(3.4.2) \quad a_i \leq nd/2, \quad b_i \geq nd/2 + 1 \quad (i \in [1, d-r]).$$

**3.5. Remark.** There are many examples with  $M' = 0$  other than Example (3.1). For instance, if  $f = xyz$  with  $n = d = 3$ , then

$$\mu_k = \mu''_k = \begin{cases} 3 & \text{if } k > 3, \\ 1 & \text{if } k = 3, \\ 0 & \text{if } k < 3, \end{cases} \quad \nu_k = \begin{cases} 3 & \text{if } k > 6, \\ 2 & \text{if } k = 6, \\ 0 & \text{if } k < 6. \end{cases} \quad \gamma_k = \begin{cases} 3 & \text{if } k = 4 \text{ or } 5, \\ 1 & \text{if } k = 3 \text{ or } 6, \\ 0 & \text{if } k < 3 \text{ or } k > 6. \end{cases}$$

In this case, the spectrum of  $f$  (see [St2]) seems to be given by the Poincaré series of the kernel and the cokernel of the complex

$$d : N \rightarrow M,$$

up to the replacement of  $t$  with  $t^{1/3}$ , where the differential  $d$  is induced by the differential of the de Rham complex  $(\Omega^\bullet, d)$ , and preserves the grading up to the shift by 3. (Some details may be given in a forthcoming paper.)

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